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Purely chromatic and hyper chromatic lattices

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Dedicated to Professor N. K. Thakare on the occasion of his 81^{st} *birthday.*

We introduce the concepts of a hyper chromatic, a critically chromatic and a purely chromatic lattice. We obtain some characterizations for purely chromatic and hyper chromatic dismantlable lattices. We prove relationships between the chromatic number of two lattices and their linear sum, vertical sum and adjunct.

Keywords: Purely chromatic; critically chromatic and hyper chromatic posets; vertical sum; linear sum; adjunct and dismantlable lattice.

AMS Subject Classification: 06A07, 05C15, 06E05, 06E10

1. Introduction and Preliminaries

Many researchers have proved results by associating a graph with a lattice. For example Bollobás [1], and Rival [9] studied the covering graph of a lattice. Filipov [3] studied the comparability graph, Nimbhorkar *et al.* [6] introduced the zero divisor graph of a lattice and many other researchers have associated different types of graphs with lattices. In this paper, we consider the covering graph of a lattice. Throughout in this paper, all lattices, posets are finite and all graphs are finite and simple (a graph is called simple, if its edge set contains neither a loop nor multiple edges i.e. edges connecting the same pair of vertices).

Let G be a graph and $V(G)$ and $E(G)$ denote its vertex set and the edge set, respectively. The chromatic number of the graph G, denoted by $\chi(G)$, is the minimum number of colors needed to color the vertices so that adjacent vertices receive different colors. *An independent set* in a graph is a set of pairwise nonadjacent vertices. So for any graph G the chromatic number $\chi(G)$ is the minimum number of independent sets needed to partition $V(G)$.

Any finite poset P can be visualized as a graph. Given a poset P, the *covering graph* of P, is the graph $G(P)$, whose vertex set is P and $\langle x, y\rangle$ is an edge if either x covers y (denoted by $y \prec x$) or y covers x. Let us denote the vertex set and the edge set of $G(P)$ by $V(P)$ and $E(P)$, respectively.

A poset P is said to be *k-chromatic* if $\chi(P) = \chi(G(P)) = k$ and k-colorable if $\chi(P) \leq k$. It is known that for every positive integer k there is a poset P which is k-chromatic.

In a conference on lattice theory held at Szeged in 1974, Burmeister and Rival posed the following conjecture.

Conjecture. *The graph of a lattice is always* 3*-colorable.*

Bollobás [1] rejected this conjecture by proving the following theorem.

Theorem 1.1. *Given a natural number k there is a lattice L whose covering graph $G(L)$ is not k-colorable.*

We recall the concept of a dismantlable lattice which is introduced by Rival [9].

Definition 1.2. A finite lattice L with n elements is called a dismantlable lattice if there exists a chain $L_1 \subset L_2 \subset \cdots \subset L_n (= L)$ of sublattices $L_i, (i = 1, 2, \ldots, n)$ of L such that $|L_i| = i$ for each i.

Clearly, a sublattice of a dismantlable lattice is also dismantlable. All the lattices, shown in Fig. 1 are dismantlable. Kelly and Rival [5] proved a characterization theorem and Thakare *et al.* [10] proved a structure theorem for this class of lattices.

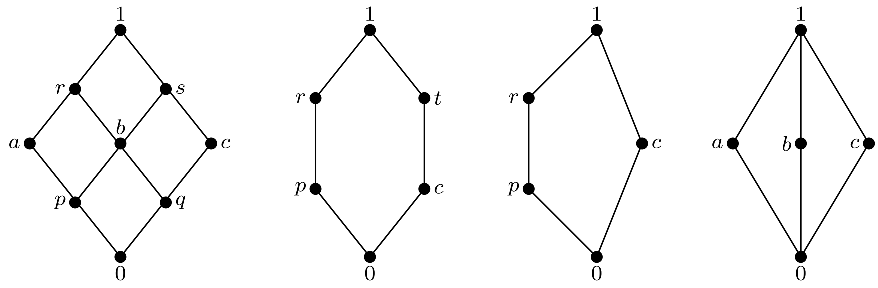
Pawar and Bhamre [7] have recently established a characterization theorem for 2-chromatic dismantlable lattices and gave chromatic classification of the class of dismantlable lattices. The following results are from [7].

Theorem 1.3. (i) *A finite semi-modular lattice is* 2*-chromatic.*

(ii) *A dismantlable lattice is* 3*-colorable.*

(iii) *A dismantlable lattice is* 2*-chromatic if and only if lengths of all maximal chains in it are of the same parity* (*i.e. all maximal chains are of even length or all of them are of odd length*).

From these results, we conclude that all posets having less than five elements are 2-colorable and the poset N_5 is 3-chromatic. Thus, every proper subposet of N_5 is 2-colorable. Consider the lattice O_6 in the Fig. 1, we observe that N_5 is a



$L_1 = \mathcal{C}_3 \times \mathcal{C}_3$ $\quad O_6 \quad N_5 \quad M_3$

Fig. 1. Some dismantlable lattices.

sublattice of $O_6, \chi(O_6) = 2$ and $\chi(N_5) = 3$. This indicates that, 'a lattice L may have a sublattice L' such that $\chi(L') > \chi(L)$'.

This observation motivates us to introduce the concepts of a purely chromatic, a critically chromatic and a hyper chromatic poset as well as a lattice as follows.

Definition 1.4. A poset P (lattice L) is said to be a *purely chromatic poset* (*lattice*) if for every subposet Q of P (sublattice L' of L), $\chi(Q) \leq \chi(P)$ ($\chi(L') \leq \chi(L)$). A poset P is called a *critically chromatic poset* if for every proper subposet Q of P, $\chi(Q) < \chi(P)$ and a *hyper chromatic poset* if there exists a subposet Q of P, such that $\chi(Q) > \chi(P)$.

Similarly, we can define a *critically chromatic lattice* and a *hyper chromatic lattice*.

The work in [1, 10] and [7] motivated us to define and study purely chromatic, hyper chromatic and critically chromatic posets.

The operations such as the linear sum, the vertical sum, the horizontal sum and adjunct, etc. on lattices (posets) are studied by many researchers.

In this paper, we obtain relationships between the chromatic numbers of two lattices (posets) and their linear sum, vertical sum and adjunct. Some characterizations for purely 2-chromatic lattices and dismantlable hyper chromatic lattices are also proved.

The undefined terms and notations from lattice theory are from Davey and Priestley [2] or Grätzer [4] and from graph theory are from West [11].

2. Coloring of Posets

In this section, we show that a dismantlable lattice is a hyper chromatic lattice if and only if it is 2-chromatic and nonmodular. Throughout in this paper, we denote a chain with n elements by $\mathcal{C}_n$.

Obviously, every critically chromatic poset is purely chromatic but not conversely, e.g. $\mathcal{C}_3$ is purely 2-chromatic but not critically chromatic. In fact, $\mathcal{C}_2$ is the only critically 2-chromatic poset. Note that, if $Q \subset P$ and $G(Q)$ is a subgraph of $G(P)$, then $\chi(Q) \leq \chi(P)$. Thus, P is hyper chromatic if and only if there exists a sub-poset Q of a poset P such that $G(Q)$ is not a subgraph of $G(P)$. As every sublattice of a lattice L is a sub-poset of L, we have the following results.

Proposition 2.1. *If a poset P forms a lattice and it is a purely chromatic poset, then P is also a purely chromatic lattice.*

Proposition 2.2. *If a lattice L is a hyper chromatic lattice, then L is also a hyper chromatic poset.*

We prove the following characterization.

Theorem 2.3. *A nontrivial lattice of finite length is a purely 2-chromatic lattice if and only if it is modular.*

Proof. It is well known that a lattice is monochromatic if and only if it is a trivial lattice. By Theorem 1.3, every nontrivial modular lattice is 2-chromatic. Since every sublattice of a modular lattice is modular, it follows that a nontrivial modular lattice is a purely 2-chromatic lattice.

Conversely, let L be a purely 2-chromatic nontrivial lattice of finite length. Then $\chi(L) = 2$. If L' is a sublattice of L, then $\chi(L') \leq 2$. Since $\chi(N_5) = 3$, L cannot have a sublattice isomorphic to N_5. Hence, L is modular. □

The following example shows that the converse of Proposition 2.1 and that of Proposition 2.2 does not hold.

Example 2.4. Let us consider the lattices L_1 and N_5 as depicted in Fig. 1, we observe that $\chi(L_1) = 2$ and $\chi(N_5) = 3$. We note that L_1 is a modular lattice. Since every sublattice of a modular lattice is modular, by Theorem 2.3 every sublattice of L_1 is 2-colorable. Thus, L_1 is a purely chromatic lattice.

Although N_5 is not a sublattice of L_1, it is a sub-poset of L_1. Thus, L_1 is a hyper chromatic poset i.e. it is not a purely chromatic poset.

Thus from Theorem 2.3 and Example 2.4, we have the following remark.

Remark 2.5. A modular lattice is a purely chromatic lattice but it may be a hyper chromatic poset.

Corollary 2.6. *A dismantlable lattice L is a purely chromatic lattice if and only if $\chi(L) = 3$ or L is 2-colorable and modular.*

Proof. Since a sublattice of a dismantlable lattice is dismantlable, the result follows from Theorem 1.3(ii) and 2.3. □

Now we characterize dismantlable hyper chromatic lattices as follows.

Theorem 2.7. *A dismantlable lattice is a hyper chromatic lattice if and only if it is* 2*-chromatic and nonmodular.*

Proof. Let L be a 2-chromatic, nonmodular lattice. Then there is a sublattice L' of L which is isomorphic to N_5 and hence $\chi(L') = 3$. Thus, L is a hyper chromatic lattice.

Conversely, let L be a dismantlable, hyper chromatic lattice. As a trivial lattice is the only 1-chromatic lattice, $\chi(L) \neq 1$. Let if possible $\chi(L) = 3$. By Theorem 1.3(ii), for every sublattice L' of L, $\chi(L') \leq 3$, which contradicts the fact that L is hyper chromatic. Hence, $\chi(L) = 2$ and nonmodularity of L follows from Theorem 2.3. □

Remark 2.8. Example 2.4 shows that a dismantlable, 2-chromatic and modular lattice may be a hyper chromatic poset.

3. Operations on Lattices and Coloring

In this section, we recall some known operations on posets such as the linear sum (also called an ordinal sum), the vertical sum, the horizontal sum etc. and study them in the context of coloring of lattices.

The following definition is from Davey and Priestley [2, p. 17].

Definition 3.1. Let P and Q be disjoint posets. The *linear sum or ordinal sum* of P and Q, denoted by $P \oplus Q$, is defined by taking the following order relation on $P \cup Q : x \leq y$ if and only if $x, y \in P$ and $x \leq y$ in P, or $x, y \in Q$ and $x \leq y$ in Q, or $x \in P$ and $y \in Q$.

The *vertical sum* of two bounded posets P and Q is obtained from the linear sum $P \oplus Q$ by identifying the greatest element of P with the least element of Q. The *horizontal sum* of bounded posets P and Q is obtained from their disjoint union $P \cup Q$ by identifying the greatest elements of the two posets and also identifying the least elements. These two concepts are from Davey and Priestley [2, p. 84]. The concept of a vertical sum of two posets can be defined more explicitly as follows.

Definition 3.2. Let P_1 be a poset with the largest element and P_2 be a poset with the least element such that the greatest element of P_1 and the least element of P_2 are the same, say α, and $P_1 \cap P_2 = \{\alpha\}$, then the vertical sum of P_1 with P_2, denoted by $P_1 \circ P_2$, is a poset $(P_1 \cup P_2, \leq)$, where $x \leq y$ if and only if $x, y \in P_1$ and $x \leq y$ in P_1 or $x, y \in P_2$ and $x \leq y$ in P_2 or x in P_1 and y in P_2.

Using these two definitions, we have:

Proposition 3.3. *Let L_1 and L_2 be two lattices.*

(1) *Let L' be a sublattice of $L_1 \oplus L_2$. If both $L' \cap L_1$ and $L' \cap L_2$ are nonempty, then $L' = (L' \cap L_1) \oplus (L' \cap L_2)$.*

(2) *Let L' be a sublattice of $L_1 \circ L_2$. If both $L' \cap L_1$ and $L' \cap L_2$ are nonempty, then $L' = (L' \cap L_1) \circ (L' \cap L_2)$ or $L' = (L' \cap L_1) \oplus (L' \cap L_2)$.*
(3) *$L_1 \oplus L_2$ is distributive(modular) if and only if both L_1 and L_2 are distributive(modular).*
(4) *$L_1 \circ L_2$ is distributive(modular) if and only if both L_1 and L_2 are distributive(modular).*

Thakare *et al.* [10] introduced the concept of an adjunct of lattices. We extend it to posets as follows.

Definition 3.4. Let P and Q be two disjoint finite posets and (a, b) be a pair of distinct elements in P such that $a < b$ and a is not covered by b. We define the partial order $\leq$ on $P \cup Q$ with respect to the pair (a, b) as follows: $x \leq y$ in $P \cup Q$ if and only if $x, y \in P$ and $x \leq y$ in P or $x, y \in Q$ and $x \leq y$ in Q or $x \in P, x \leq a$ in P and $y \in Q$ or $x \in Q$, $y \in P$ and $b \leq y$ in P. The procedure of obtaining a poset $(P \cup Q, \leq)$ in this way is called the adjunct operation of P with Q at (a, b) and the new poset is denoted by $P]_a^b Q$.

The diagram of $P]_a^b Q$ is obtained by placing the diagram of P and the diagram of Q side by side in such a way that the maximal elements of Q are at lower position than that of b and the minimal elements of Q are at higher position than that of a and then we add the edges connecting every maximal element of Q with b and every minimal element of Q with a. It is easy to note that, if L_1 and L_2 are two lattices, then $L_1]_a^b L_2$ is also a lattice and L_1 and L_2 are its sublattices. The lattice L_2 as depicted in Fig. 2, illustrates $\mathbf{2^3}]_c^1 \mathcal{C}_1$.

In particular, if P and Q are bounded posets, then $|E(P \oplus Q)| = |E(P)| + |E(Q)| + 1, |E(P \circ Q)| = |E(P)| + |E(Q)|$ and $|E(P]_a^b Q)| = |E(P)| + |E(Q)| + 2$. Due to this fact the vertical sum and the adjunct of two lattices are respectively called 0-*sum* and 2-*sum*; see Pawar and Waphare [8]. Apparently, one may feel that

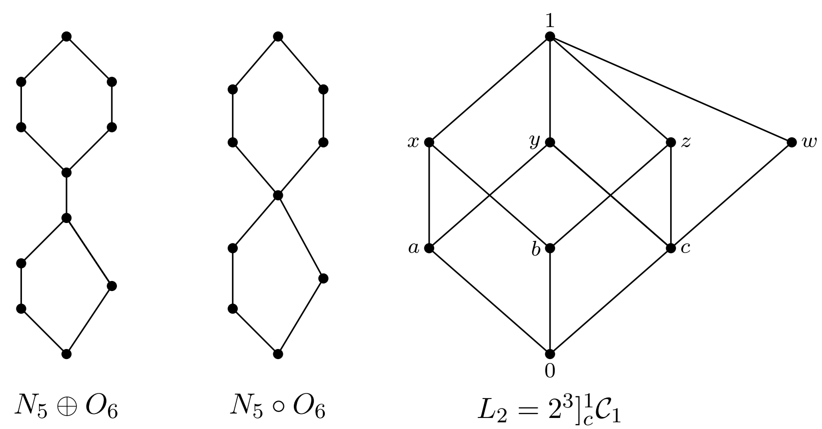


Fig. 2. Linear sum, vertical sum and adjunct of lattices.

the horizontal sum of two lattices L_1 and L_2 is a special case of the adjunct of the two lattices L_1 and L_2 at $(0,1)$, but that is not true. Because, $|L_1]_0^1 L_2| = |L_1|+|L_2|$, whereas the number of elements in the horizontal sum of L_1 and L_2 is $|L_1|+|L_2|-2$. Thus, the adjunct of two bounded posets is a more general concept than that of their horizontal sum. After introducing the concept of an adjunct of two lattices, Thakare *et al.* [10] established a structure theorem for dismantlable lattices as follows.

Theorem 3.5 (Structure theorem). *A finite lattice is dismantlable if and only if it is an adjunct of chains. i.e. a lattice L is dismantlable if and only if $L = C_1]_{a_1}^{b_1} C_2]_{a_2}^{b_2} \cdots]_{a_{k-1}}^{b_{k-1}} C_k$, where $C_1, C_2, \ldots, C_k$ are disjoint chains in L.*

Note that a representation of a dismantlable lattice as an adjunct of chains is not unique. However, the number of chains in any adjunct representation of a dismantlable lattice remains the same. Regarding coloring of these operations, we have the following results.

Theorem 3.6. *Let P and Q be two bounded posets and $m = \mathrm{Max}\{\chi(P), \chi(Q)\}$.*

(1) *If P and Q are disjoint, then*

$$\chi(P \oplus Q) = \begin{cases} 2 & \text{if } \chi(P) = \chi(Q) = 1 \\ m & \text{otherwise.} \end{cases}$$

(2) *If $P \cap Q = \{a\}$, where a is the greatest element of P and also the least element of Q, then $\chi(P \circ Q) = m$.*

Proof. Let $\chi(P) = k_1$ and $\chi(Q) = k_2$ and $m = \max\{k_1, k_2\}$. Consider partitions, $\{V_1, V_2, \ldots, V_{k_1}\}$ and $\{V'_1, V'_2, \ldots, V'_{k_2}\}$ of independent subsets of P and Q, respectively such that the greatest element 1_P of P is in V_1 and the least element 0_Q of Q is in V'_1. Also, $[m] = \{1, 2, \ldots, m\}$ be the set of m colors.

(1) Let P and Q be disjoint posets. If $\chi(P) = \chi(Q) = 1$, then $P \oplus Q$ is a chain $\mathcal{C}_2$. Hence $\chi(P \oplus Q) = 2$. Otherwise, $G(P)$ and $G(Q)$ are subgraphs of $G(P \oplus Q)$, therefore $\chi(P \oplus Q) \geq m$. Moreover, $E(P \oplus Q) = E(P) \cup E(Q) \cup \{\langle 1_P, 0_Q \rangle\}$. Therefore, a coloring function $c : P \oplus Q \to [m]$ defined as

$$c(x) = \begin{cases} i & \text{if } x \in V_i \quad i = 1, 2, \ldots, k_1, \\ j + 1 (\text{mod } m) & \text{if } x \in V'_j \quad j = 1, 2, \ldots, k_2 \end{cases}$$

is a proper coloring of L. This leads to $\chi(P \oplus Q) = m$.

(2) If $1_P = 0_Q = a$, then $P \circ Q$ is defined. Using the same techniques as used in (1), we observe that, a coloring function $c' : P \circ Q \to [m]$ defined as

$$c'(x) = i \quad \text{if } x \in V_i \text{ and/or } x \in V'_i i = 1, 2, \ldots, m,$$

is a proper coloring of L. This leads to $\chi(P \circ Q) = m$. ☐

The following corollary follows from Proposition 3.3 and Theorem 3.6.

Corollary 3.7. *The linear sum of two purely chromatic lattices is a purely chromatic lattice. Similarly, the vertical sum of two purely chromatic lattices is a purely chromatic lattice.*

The following examples show that, the converse of both the statements in Corollary 3.7 do not hold.

Example 3.8. The lattices $N_5 \oplus O_6$ and $N_5 \circ O_6$ depicted in Fig. 2 are purely chromatic but O_6 is not purely chromatic.

Example 3.9. The lattice O_6 depicted in Fig. 1 is a horizontal sum of two disjoint copies of $\mathcal{C}_4$. Here, $\mathcal{C}_4$ is purely chromatic lattice but O_6 is not.

This observation leads to the computation of the chromatic number of an adjunct of two lattices as follows.

Let L be a lattice. We denote by $l(L)$, the length of a longest chain in L.

Theorem 3.10. *If L_1 and L_2 are two lattices, $L = L_1]_a^b L_2$ and $m = \mathrm{Max}\{\chi(L_1), \chi(L_2)\}$, then*

$$\chi(L) = \begin{cases} m+1 & \text{if } m = 2 \text{ and } l([a,b]) \text{ and } l(L_2) \text{ are not of the same parity} \\ m & \text{otherwise.} \end{cases}$$

Proof. Let $L = L_1]_a^b L_2$, $\chi(L_1) = k_1$ and $\chi(L_2) = k_2$ and $m = \max\{k_1, k_2\}$. As $G(L_1)$ and $G(L_2)$ are subgraphs of $G(L)$, clearly, $\chi(L) \geq m$. It follows from the Definition 3.4 that $k_1 \geq 2$. Hence, $m \geq 2$ and the lattice L_2 has the least element 0 and the greatest element 1. Also, the edge set of $G(L)$ is partitioned into three sets $E(L_1), E(L_2)$ and $\{\langle a, 0\rangle, \langle 1, b\rangle\}$. To determine $\chi(L)$, consider the partitions $\{V_1, V_2, \ldots, V_{k_1}\}$ and $\{V_1', V_2', \ldots, V_{k_2}'\}$ of independent subsets of L_1 and L_2, respectively and a set of m colors is denoted by $[m] = \{1, 2, \ldots, m\}$ of m colors.

Case-I: Let $m = 2$. Suppose that $l([a,b])$ and $l(L_2)$ are of the same parity.

If $l([a,b])$ and $l(L_2)$ both are even, then every saturated chain connecting a and b is of even length. Hence both a, b acquire the same color. Similarly, 0 and 1 also acquire the same color. Hence, without loss of generality, assume that $a, b \in V_1$ and $0, 1 \in V_1'$. Then we observe that, $c : L \to [m]$ defined by

$$c(x) = \begin{cases} 1 & \text{if } x \in V_1 \text{ or } x \in V_2', \\ 2 & \text{if } x \in V_2 \text{ or } x \in V_1' \end{cases}$$

is a proper coloring of L.

If $l([a,b])$ and $l(L_2)$ both are odd, then every saturated chain connecting a and b is of odd length, hence both a, b acquire different colors. Similarly, 0 and 1 also acquire different colors. Hence, without loss of generality, assuming that

$a \in V_1, b \in V_2, 0 \in V_1'$ and $1 \in V_2'$, it can be verified that, the same color function c defined above is a proper coloring of L. Thus $\chi(L) = 2$.

Case-II: Let $m = 2$. Suppose that $l([a, b])$ and $l(L_2)$ are not of the same parity.

If $l[a, b]$ is even and $l(L_2)$ is odd, then there exist two saturated chains, $C_1 \subset L_1$ connecting a and b and $C_2 \subset L_2$ connecting 0 and 1 such that $l(C_1)$ is even and $l(C_2)$ is odd. Moreover, $C_1]_a^b C_2$ is a sublattice of L. As $G(C_1]_a^b C_2)$ is subgraph of $G(L)$ and it is an odd cycle and $\chi(C_1]_a^b C_2) = 3$. Therefore $\chi(L) \geq 3, a, b$ acquire the same color and 0 and 1 acquire different colors. Without loss of generality, assume that $a, b \in V_1, 0 \in V_1'$ and $1 \in V_2'$, then we observe that $c' : L \to [3]$ defined as

$$c'(x) = \begin{cases} 1 & \text{if } x \in V_1, \\ 2 & \text{if } x \in V_2 \text{ or } x \in V_1', \\ 3 & \text{if } x \in V_2' \end{cases}$$

is a proper coloring of L. Thus $\chi(L) = 3$.

Similarly, it can be shown that, if $l([a, b])$ is odd and $l(L_2)$ is even then $\chi(L) = 3$.

Case-III: Let $m \geq 3$. Depending on $l([a, b])$ and $l(L_2)$, without loss of generality, assuming that $a, b \in V_1$ or $a \in V_1$ and $b \in V_2$ as well as $0, 1 \in V_1'$ or $0 \in V_1'$ and $1 \in V_2'$, we observe that, in each case a coloring function $c'' : L \to [m]$ defined as

$$c''(x) = \begin{cases} i & \text{if } x \in V_i \quad i = 1, 2, \ldots, k_1 \\ j + 1 \pmod{m} & \text{if } x \in V_j' \quad j = 1, 2, \ldots, k_2 \end{cases}$$

is a proper coloring of L. This leads to $\chi(L) = m$.

The above three cases complete the proof. □

We have the following theorem for dismantlable lattices.

Theorem 3.11. *Let L_1 and L_2 be two dismantlable lattices,*

(1) *If $L_1 \cap L_2 = \emptyset$, then $L_1 \oplus L_2$ is a purely chromatic lattice if and only if both L_1 and L_2 are modular or at least one of L_1 and L_2 is 3-chromatic.*
(2) *$L_1 \circ L_2$ is a purely chromatic lattice if and only if both L_1 and L_2 are modular or at least one of L_1 and L_2 is 3-chromatic.*
(3) *If $L_1 \cap L_2 = \emptyset$, then $L_1 \oplus L_2$ is a hyper chromatic lattice if and only if both L_1 and L_2 are 2-chromatic and at least one of L_1 and L_2 is nonmodular.*
(4) *If $L_1 \cap L_2 = \{a\}$, where a is the greatest element of L_1 and the least element of L_2. Then $L_1 \circ L_2$ is a hyper chromatic lattice if and only if both L_1 and L_2 are 2-chromatic and at least one of L_1 and L_2 is nonmodular.*

Proof. As L_1 and L_2 are dismantlable lattices, by Theorem 3.5, we can write $L_1 = C_0^1]_{a_1}^{b_1} C_1^1]_{a_2}^{b_2} \cdots]_{a_k}^{b_k} C_k^1$ and $L_2 = C_0^2]_{a_1'}^{b_1'} C_1^2]_{a_2'}^{b_2'} \cdots]_{a_{k'}'}^{b_{k'}'} C_{k'}^2$. If $L_1 \cap L_2 = \emptyset$, then

$L_1 \oplus L_2$ exists. Also $C_0^1 \oplus C_0^2$ is a maximal chain in $L_1 \oplus L_2$. Moreover, we can express $L_1 \oplus L_2$ as

$$L_1 \oplus L_2 = (C_0^1 \oplus C_0^2)]_{a_1}^{b_1} C_1^1]_{a_2}^{b_2} \cdots]_{a_k}^{b_k} C_k^1]_{a_1'}^{b_1'} C_1^2]_{a_2'}^{b_2'} \cdots]_{a_{k'}'}^{b_{k'}'} C_{k'}^2.$$

Therefore, by Theorem 3.5, $L_1]_a^b L_2$ is also a dismantlable lattice.

Similarly, we can prove that $L_1 \circ L_2$ (if exists) is also dismantlable.

Now, (1) and (2) follow from Theorem 1.3 and Corollary 2.6. Also, (3) and (4) can be proved by using Theorems 1.3 and 2.7. □

Remark 3.12. In Example 3.9, the lattice O_6 is not purely chromatic but it can be expressed as an adjunct of two purely chromatic dismantlable lattices namely, $\mathcal{C}_4]_0^1\mathcal{C}_2$. The lattice $L_2 = \mathbf{2^3}]_c^1\mathcal{C}_1$ as depicted in Fig. 2 is an adjunct of two purely chromatic lattices but it is not a purely chromatic lattice. Moreover, it is an adjunct of two distributive lattices but it is neither distributive nor modular.

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